IMPACTS OF STRUCTURAL PERTURBATIONS ON THE SYNCHRONIZABILITY OF DIFFUSIVE NETWORKS

Camille Poignard, Jan Philipp Pade and Tiago Pereira

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Introduction

We study the effects of structural graphs perturbations of diffusive directed networks on their ability to synchronize: that is to say, we look at the hinderance or reinforcement of synchronizability when we add directed links of small weights to a directed network or when we perturb the weights of some of its existing links.
Analogous results to those presented here can be found in the undirected setting using Fiedler theory. The reader who would like to know more about it is referred to [2] for more details.

One could believe that adding new links in a network reinforce its synchronizability. This has been shown to be wrong in [4], even for the situation where a weakly connected network is transformed into a strongly connected one as illustrated in Figure 1 with a network with 5 nodes. This figure presents

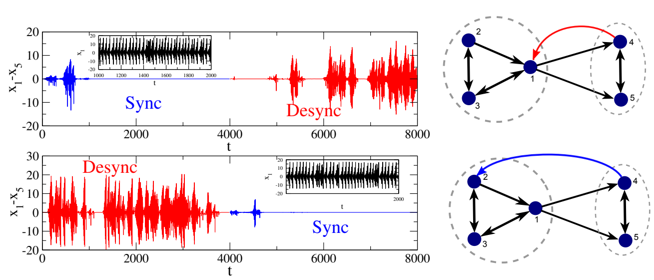

Figure 1: Improving the network structure can increase or decrease the synchronizability of a network. Here we pass from a weakly connected network to a strongly connected one by adding one link.

a *master-slave* configuration as the subnetwork consisting of nodes 1, 2 and 3 drives the subnetwork consisting of nodes 4 and 5. An interesting point to notice is that hindering synchronization is not about breaking this configuration as shown in the second graph of this figure (where the adding of a link makes the network synchronizes).

Results

The models: To a directed graph (or digraph) we can associate a dynamical network with diffusive coupling, i.e:

$$\dot{x}_i = \boldsymbol{f}(x_i) + \alpha \sum_{j=1}^{N} W_{ij} \boldsymbol{H}(x_j - x_i) \qquad i = 1, 2, \dots, N, \tag{1}$$

where N is the number of nodes of the network, α is the overall coupling strength, $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^m$ is the local dynamics of each node and $\boldsymbol{H} : \mathbb{R}^m \to \mathbb{R}^m$ is the coupling function between the nodes.
The synchronization of the network given by (1) corresponds to the asymptotical convergence towards the diagonal in $\mathbb{R}^{Nm}$ i.e. to the set $M := \{x_i \in \mathbb{R}^m \text{ for } i \in \{1, \cdots, n\} : x_1 = \cdots = x_n\}$. The rigorous statement of exponentially fast convergence under very mild conditions has been established in [3]: the authors proved that for a network of which underlying graph admits a diverging spanning tree and for which the local dynamics $\boldsymbol{f}$ is bounded in a compact set of $\mathbb{R}^m$, then under mild conditions on $D\boldsymbol{H}(0)$, there exists a critical coupling $\alpha_c(\lambda_2(L), \boldsymbol{f}, \boldsymbol{H})$ such that for any $\alpha > \alpha_c(\lambda_2(L), \boldsymbol{f}, \boldsymbol{H})$ the diagonal M attracts uniformly each trajectory in an open neighborhood of M in $\mathbb{R}^{Nm}$.

Using a structural genericity property of graph Laplacians spectra

From the result proved in [3] mentioned above, we deduce that for a fixed $\boldsymbol{f}, \boldsymbol{H}$, the synchronizability of a network fully depends on the spectral gap $\lambda_2(L)$ (which is non null since $\mathscr{G}$ admits a diverging spanning tree) of its associated Laplacian matrix L. This leads us to introduce the following definition:

Definition 1. *We say that the network $(\mathscr{G}_1, \boldsymbol{f}, \boldsymbol{H})$ is more synchronizable than $(\mathscr{G}_2, \boldsymbol{f}, \boldsymbol{H})$ if we have: $\lambda_2(L_1) > \lambda_2(L_2)$, where L_1 denotes the Laplacian matrix of $\mathscr{G}_1$ and L_2 the one of $\mathscr{G}_2$.*

From this definition, our problem now consists in understanding how the spectral gap evolves under structural perturbations of a given network. To perform this analysis, we need to work under the assumption that the spectrum of the corresponding Laplacian matrix is simple. We have [1]:

Theorem 0.1 (C.P. et al. 2017)**.** *Let $\mathcal{G}$ a weighted weakly connected digraph admitting a diverging spanning tree. Then, for a generic choice of the weights of the existing edges of $\mathcal{G}$, the eigenvalues of its Laplacian matrix L are all simple.*

In [1] we have described several such structural genericity properties for graph Laplacians not only for directed but also for undirected graphs.

A result on the impacts of structural perturbations on the synchronizability of networks

From this spectral structural genericity result, we can now establish our main result on the dynamical impacts of adding links. We focus on networks presenting a master-slave configuration:

$$L = \begin{pmatrix} L_1 & 0 \\ -C & L_2 + D_C \end{pmatrix}, \tag{2}$$

where $L_1 \in \mathbb{R}^{N\times N}$ and $L_2 \in \mathbb{R}^{m\times m}$ are the respective Laplacians of the strong components, $C \in \mathbb{R}^{m\times N}$ is the adjacency matrix of the cutset and D_C is again a diagonal matrix with the row sums of C on its diagonal.
Given a graph Laplacian L as in Eq. (2), a structural modification in opposite direction of the cutset induced by $\Delta \in \mathbb{R}^{m\times N}$ corresponds to:

$$L_p(\Delta) = \begin{pmatrix} L_1 + D_\Delta & -\Delta \\ -C & L_2 + D_C \end{pmatrix}.$$

Notation:
Given a Laplacian matrix L of a digraph with simple spectral gap $\lambda_2(L)$ and a nonnegative matrix Δ we set: $s(\Delta) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\lambda_2\left(L_p(\varepsilon\Delta)\right) - \lambda_2(L)\right)$ the rates of change of the spectral gap map under the small perturbations $\varepsilon\Delta$.

With this notation, $s(\Delta) > 0$ means an enhancement of synchronisation under a structural perturbation given by Δ, and $s(\Delta) < 0$ corresponds to a decrease of synchronisation.
Now our final result can be stated in the following way:

Theorem 0.2 (C.P. et al. 2017)**.** *Let a directed graph $\mathcal{G}$ consist of two strong components connected by a cutset with adjacency matrix C and write the associated Laplacian as in (2): Assume that $\mathcal{G}$ admits a diverging spanning tree and that the spectral gap λ_2 is an eigenvalue of $L_2 + D_C$. Then, for a generic choice of the nonzero weights of L we have:*

(i) Improving synchronizability by reinforcing the cutset *If λ_2 is an eigenvalue of $\boldsymbol{L}_2 + \boldsymbol{D}_C$, then the network's synchronizability increases for arbitrary structural perturbations $\boldsymbol{\Delta}$ in direction of the cutset.*

(ii) Non-optimality of master-slave configurations *Assume λ_2 is an eigenvalue of $\boldsymbol{L}_2 + \boldsymbol{D}_C$. Then we have the following statements:*

(a) *There exists a structural perturbation $\boldsymbol{\Delta}$ in opposite direction of the cutset such that $s(\boldsymbol{\Delta}) > 0$.*

(b) *There exists a constant $\delta(\boldsymbol{L}_1) > 0$ and at least one node $1 \le k_0 \le n$ (in the driving component) such that if we have $0 < \lambda_2 < \delta(\boldsymbol{L}_1)$, then $s(\boldsymbol{\Delta}) > 0$ for any structural perturbation $\boldsymbol{\Delta}$ consisting of only one link in opposite direction of the cutset and ending at node k_0.*

(c) *If moreover $\boldsymbol{L}_1$ has zero column-sums, then there exists a constant $\delta(\boldsymbol{L}_1) > 0$ such that if $0 < \lambda_2 < \delta(\boldsymbol{L}_1)$ we have $s(\boldsymbol{\Delta}) > 0$ for any structural perturbation $\boldsymbol{\Delta}$ in opposite direction of the cutset.*

(iii) Hindering synchronizability by breaking the M-S configuration *There exists a cutset $\boldsymbol{C}$ for which λ_2 is an eigenvalue of $\boldsymbol{L}_2 + \boldsymbol{D}_C$ and a perturbation $\boldsymbol{\Delta}$ in opposite direction of $\boldsymbol{C}$ such that: if $\boldsymbol{L}_1$ admits a positive eigenvalue sufficiently small then we have $s(\boldsymbol{\Delta}) \le 0$.*

The generalization of Theorem 0.2 to a higher number $p \ge 2$ of strongly connected components $\boldsymbol{L}_1, \cdots, \boldsymbol{L}_p$ is (provided the master-slave configuration is respected), a straightforward generalization. We refer the reader to [1] for a similar theorem in the case of directed structural perturbations of weighted undirected graphs.

References

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