DIFFUSION ON MULTIPLEX NETWORKS WITH ASYMMETRIC COUPLING

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Abstract

A leading approach for studying biological, technological, and social networks is to represent them by multiplex networks in which different layers encode different types of connections and/or interacting systems [4]. Examples include interconnected critical infrastructures such as transportation systems, power grids, and water lines [3] as well as multimodal social networks containing different categories of social ties [5]. Here, we study diffusion processes on multiplex networks that are allowed to contain directed edges within and/or between layers. We develop perturbation theory to understanding the effects of directedness in the multiplex setting, and our work extends existing theories that are either restricted to undirected multiplex networks [2, 7], or which allow directed edges only within, but not between, layers [9].

Background Information

Our work is based on studying the eigenvalues and eigenvectors of Laplacian matrices, which are widely used to study dynamics on networks [6,8] and also provide theoretical foundations for many machine-learning algorithms [1]. We study a generalization of Laplacian matrices for multiplex networks called *supra-Laplacians* [7],

$$\mathbb{L}(\omega) = \mathbb{L}^{\mathrm{L}} + \omega \mathbb{L}^{\mathrm{I}}, \tag{1}$$

where $\mathbb{L}^{\mathrm{L}}$ represents an *intralayer* supra-Laplacian that encodes connections within individual layers and $\mathbb{L}^{\mathrm{I}}$ represents an *interlayer* supra-Laplacian that encodes connections between layers. Parameter $\omega \geq 0$ is a coupling strength that tunes the relative diffusion rates within and between layers. The supra-Laplacian of the individual layers $\mathbb{L}^{\mathrm{L}}$ is given by

$$\mathbb{L}^{\mathrm{L}} = \bigoplus_{t=1}^{T} \mathrm{L}^{(t)} = \begin{pmatrix} \mathrm{L}^{(1)} & 0 & \cdots & 0 \\ 0 & \mathrm{L}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{L}^{(T)} \end{pmatrix}, \tag{2}$$

where each $\mathrm{L}^{(t)} = \mathrm{D}^{(t)} - \mathrm{A}^{(t)}$ is an unnormalized Laplacian matrix for layer t. That is, layer t has an adjacency matrix $\mathrm{A}^{(t)} \in \mathbb{R}^{N \times N}$ and a diagonal matrix $\mathrm{D}^{(t)}$ that encodes the nodes' intralayer degrees (i.e., degree for each node i within each layer t). We focus on *uniformly coupled* layers described by an interlayer Laplacian

$$\mathbb{L}^{\mathrm{I}} = L^{\mathrm{I}} \otimes \mathrm{I}, \tag{3}$$

where I is the identity matrix and $L^{\mathrm{I}} = D^{\mathrm{I}} - A^{\mathrm{I}}$ is an interlayer Laplacian matrix (with D^{I} and A^{I} defined similarly to D^{L} and A^{L}).

Our work builds on previous research [2,7] that analyzed the behavior of the spectrum of supra-Laplacian matrices using perturbation theory for the limits of strong and weak coupling (i.e., large and small coupling strength ω). The small and large coupling limits implement a type of time scale separation in which diffusion is much more likely to stay in the same layer (small ω) or it is much more likely to switch layers (large ω). This work, however, was restricted to undirected networks. Recently, [9] allowed for directed edges within a layer, which led to a nonmonotonic response in where there is an optimal coupling strength that is marked by a peak in λ_2, which is the second-smallest eigenvalue of $\mathbb{L}(\omega)$. This begs the question: What are other effects of directness, such as those arising when the edges between layers are directed?

Main results

Here, we extend this theory to characterize the eigenvectors and eigenvalues of supra-Laplacians for multiplex networks with more general types of coupling between layers. We extend existing research by developing theory for multiplex networks in with directed coupling within and/or between layers. This is a reasonable modeling choice because for many interconnected systems, there exists a significant asymmetry between their coupling. For example, in the case of transportation, after a person travels on an a flight between airports, they will travel by car on a road with very high probability. In contrast, after a person travels in a car, it is very unlikely that they fly on a plane. Hence, appropriate modeling for diffusion on transportation systems should allow asymmetry for the

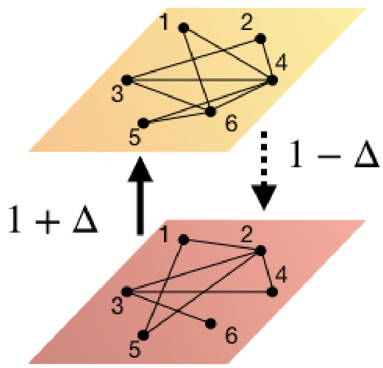

Figure 1: **Multiplex networks with asymmetric coupling between layers.** We study a toy multiplex network with $T = 2$ layers and $N = 6$ nodes that is identical to the network studied in [2], except we now allow the interlayer coupling to be asymmetric (i.e., directed) using an *asymmetry parameter* $\delta \geq 0$. Here, $\delta = 0$ recovers undirected coupling, whereas $\delta \to 1$ implements purely directed coupling.

coupling between network layers, and similar asymmetric couplings are expected to arise in myriad network-science applications ranging from brain networks to social networks.

Thus motivated, we study multiplex networks with directed edges, and we employ singular perturbation theory to rigorously characterize several asymptotic limits including the limits of strong and weak coupling between layers (similar to [2,7]), as well as a limit when the interlayer coupling points in a single direction from one layer to another. We observe new insights for how asymmetric interlayer coupling fundamentally affects diffusion on multilayer networks. We study this phenomenon for synthetic networks (e.g., Figure 1) as well as empirical network datasets such as social networks and biological networks.

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