DEGREE-TARGETED CASCADES IN MODULAR, DEGREE-HETEROGENEOUS NETWORKS

Jordan Snyder, Weiran Cai, and Raissa M. D'Souza

SIAM Workshop on Network Science 2020
July 9–10 · Toronto

Summary

We consider cascading dynamics on modular, degree-heterogeneous networks, and focus especially on the impact of seeding strategies that take node degree into account. We demonstrate that there are approximate equations (valid in the $N \to \infty$ limit) that require only one dynamical variable per module, rather than a separate variable for each degree class. These approximate equations let us prove that there is a critical level of interconnectedness between two statistically equivalent modules, below which a global cascade is impossible given initialization contained within one module, regardless of seeding strategy.

Methodological Contribution

We consider irreversible binary-state dynamics ("inactive"→"active"), where the probability that a node becomes active is a nondecreasing function of its number of active neighbors. A general technique to approximate the outcome of such dynamics subject to random seeding was developed by Gleeson [1], and rests on an assumption that the network is treelike (i.e. has few short loops); here we demonstrate that this approach can be adapted to incorporate degree-targeted seeding. A direct application of Gleeson's technique would require one dynamical variable for each pair of (module, degree) values. In contrast, we show that the dependence of activation probability on degree is entirely captured by the initial activation probability, leading to dynamics with only a single dynamical variable per module. The equations are:

$$\overline{q}^{(i)}_{n+1} = \frac{1}{\sum_j e_{ij}} \sum_j e_{ij} \left[\sum_k \frac{k}{z^{(j)}} p^{(j)}_k \left(\rho^{(j)}_{0,k} + (1-\rho^{(j)}_{0,k}) \times \right.\right.$$
$$\left.\left. \sum_{m=0}^{k-1} \binom{k-1}{m} \left(\overline{q}^{(j)}_n\right)^m \left(1-\overline{q}^{(j)}_n\right)^{k-1-m} F^{(j)}(m,k) \right) \right]$$

where e_{ij} is the fraction of links between modules i and j, $p^{(j)}_k$ is the degree distribution of module j, $z^{(j)}$ the mean degree of module j, $\rho^{(j)}_{0,k}$ the probability that a degree-k node in module j is active initially, and $F^{(j)}(m,k)$ is the *activation function*, the probability that a degree-k node in module j becomes active when it has m active neighbors. The fraction $\rho^{(i)}_n$ of nodes in module i that are active at time n is given in terms of the $\overline{q}$ variables by

$$\rho^{(i)}_n = \sum_k p^{(i)}_k \left[\rho^{(i)}_{0,k} + (1-\rho^{(i)}_{0,k}) \right.$$
$$\left. \times \sum_{m=0}^{k} \binom{k}{m} \left(\overline{q}^{(i)}_n\right)^m \left(1-\overline{q}^{(i)}_n\right)^{k-m} F^{(i)}(m,k) \right]$$

Figure 1 demonstrates that our approach accurately captures cascading dynamics for both uniform seeding (i.e. independent of degree) and seeding targeted at only the highest-degree nodes.

Conditions for Cascade Spreading

Although our theory applies to networks with an arbitrary number of modules and an arbitrary (monotone) activation function, we apply our theory to a special case that lets us draw conclusions about how cascades spread from one module to another. It consists of two-state linear threshold dynamics [2, 4] on a two-module network with a given degree distribution and fraction of intra- vs. inter-module links. By adjusting the degree distribution and the fraction of intra- vs. inter-module links, we can explore both the space of degree heterogeneity and that of modularity. This system was previously studied by Nematzadeh et al. in the case of uniform seeding [3].

We consider that all the seed nodes are contained within a single module, so that we can discern conditions under which a cascade can spread to the second module, and consider two different seeding strategies. On the one hand, we select a certain fraction of nodes uniformly at random, i.e. independently of their degree. On the other hand, we select the same total fraction of nodes but ensure that they are of the highest possible degree. We compare the two seeding strategies across the joint space of degree heterogeneity and modularity.

Results

Similarly to classic results about percolation, we find that targeting high degree nodes can lead to a global cascade in

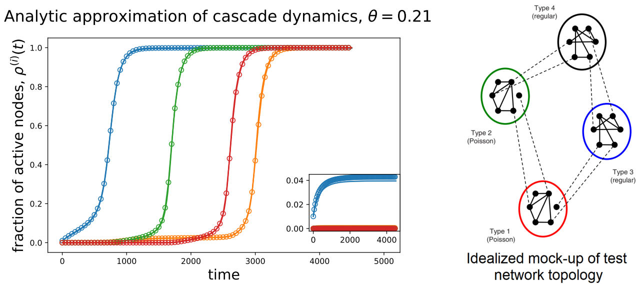

Figure 1: Degree-targeted seeding creates a global cascade (main figure) in a regime where uniform seeding does not (inset), as demonstrated by both theory (solid curves) and dynamics on an actual network of size 5×10^5 (open circles).

regimes where uniform seeding leads to only small, isolated events, and that this effect is more pronounced in more degree-heterogeneous networks. However, we also observe that regardless of the seeding protocol used, there is a critical fraction of inter-module links that must be present for a cascade started in one module to fully activate the other. These results are shown in Fig. 2.

Moreover, we support this observation by proving that it holds for the equations we derived. In essence, we prove that if the first module has more active nodes than the second, then the same is true after all nodes update their state. Hence if the cascade completely activates the second module, it must be that the first module is completely activated. Because the eventual state of the system is independent of the order in which nodes are updated, global activation is not possible unless it is possible given full activation of the first module. This defines a critical level of inter-module connectivity, μ, required for global activation to be possible. We can visualize this value of μ in terms of the existence or non-existence of a nontrivial stable fixed point of a 1d iterated map, see Fig. 3.

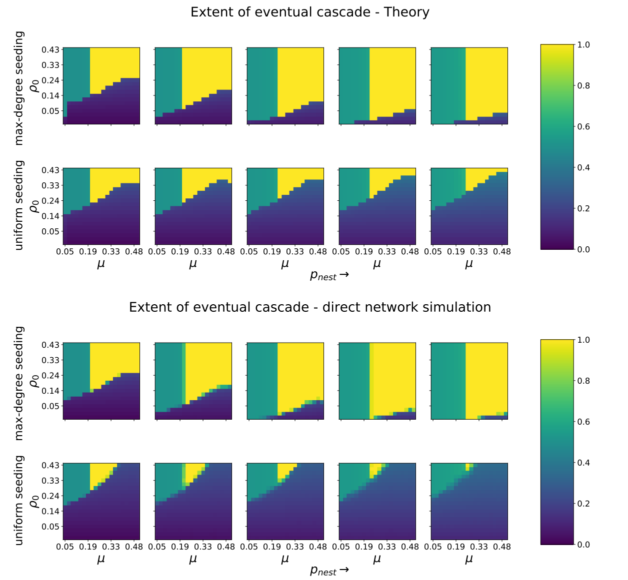

Figure 2: Summary of the joint effect of inter-module connections (μ) and degree heterogeneity (p_{nest}) on the extent of cascade spreading, under both uniform and degree-targeted seeding, as predicted by the approximation (top) and by averaging over direct simulation of the network dynamics, on networks of size $N = 2.5 \times 10^4$, averaged over ten realizations (bottom).

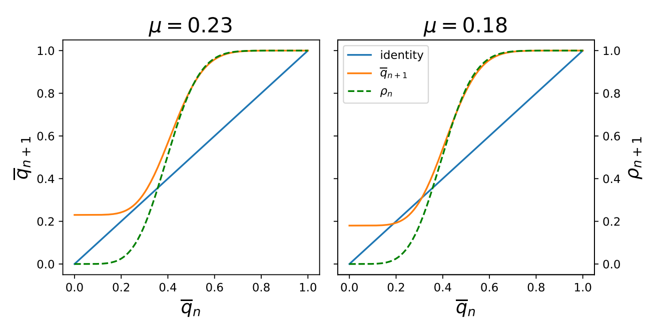

Figure 3: Visualization of the iterated map that determines the possibility of global activation, for two different values of μ. Orange curve is the right-hand side of the map while the blue line is the identity, so intersections correspond to fixed points. Notice that for $\mu = 0.23$ there is only one fixed point, at $\overline{q} = 1$, while for $\mu = 0.18$, an additional pair of fixed points (one stable, one unstable) appears with $\overline{q} < 1$, corresponding to partial activation of module 2.

References

[1] J. P. Gleeson. Cascades on correlated and modular random networks. *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics*, 77(4):1–10, 2008.

[2] M. Granovetter. Threshold Models of Collective Behavior. *American Journal of Sociology*, 83(6):1420–1443, may 1978.

[3] A. Nematzadeh, E. Ferrara, A. Flammini, and Y. Y. Ahn. Optimal network modularity for information diffusion. *Physical Review Letters*, 113(8):1–5, 2014.

[4] D. J. Watts. A simple model of global cascades on random networks. *Proc. Natl. Acad. Sci.*, 99(9):5766–5771, 2002.