NONLINEAR EIGENVECTOR CENTRALITIES

Francesco Tudisco (GSSI) and Desmond J. Higham (Edinburgh)

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Summary

Widely used centrality models rely on the Perron eigenvector of nonnegative graph matrices. These models typically require strongly connected graphs and are equivalent to defining the centrality of each node through a set of mutual reinforcing linear relations. We present a new result that allows us to use the Perron eigenvector of a much wider class of nonnegative graph operators and to consider mutual reinforcing nonlinear relations, that can for example account for higher-order interactions or multilayer data, without any requirement on the connectivity of the input network.

Eigenvector centralities

A centrality for a network G is a vector $\boldsymbol{u} \geq 0$ such that u_i quantifies the importance of the node i, according to a model that exploits only the structure of the connections in G. A larger value indicates greater importance. Further, we require $\mathbb{1}^T\boldsymbol{u} = u_1 + \cdots + u_n = 1$, so that each u_i can be interpreted as a percentage of importance.

An eigenvector centrality is a centrality defined as the eigenvector of a suitable graph matrix. For example, the *Bonacich centrality* [1] defines $\boldsymbol{u}$ as a Perron eigenvector of the adjacency matrix A, that is, a nonnegative $\boldsymbol{u}$ such that $A\boldsymbol{u} = \lambda\boldsymbol{u}$, with $\lambda > 0$ and $\mathbb{1}^T\boldsymbol{u} = 1$. Another popular example is Google's *PageRank centrality* [5], which defines $\boldsymbol{u}$ as the positive stationary distribution of the PageRank random walk transition matrix P, i.e. $P\boldsymbol{u} = \boldsymbol{u}$.

In general, we can formally associate an eigenvector centrality $M_G\boldsymbol{u} = \lambda\boldsymbol{u}$ to any entrywise nonnegative graph matrix M_G. Then, provided M_G has enough nonzero entries[1], the Perron–Frobenius (PF) theorem guarantees that the centrality $\boldsymbol{u}$ exists, it is unique and it can be efficiently computed with a power method [7].

Nonlinear eigenvector centralities

By inspecting the entrywise relation defining the Bonacich centrality $A\boldsymbol{u} = \lambda\boldsymbol{u}$, we see that this model defines u_i to be proportional to the weighted sum of the centrality scores of the neighbors of i, that is,

$$u_i \propto \sum_j A_{ij} u_j = (A\boldsymbol{u})_i. \tag{1}$$

We generalize this linear relation by means of nonlinear eigenvector centralities. For example, consider a nonnegative[2] mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\boldsymbol{x})_i = f(x_i)$ for any $\boldsymbol{x} \in \mathbb{R}^n$. If f is nonnegative, a *nonlinear Bonacich centrality* for the node i can be defined via the proportional relation

$$u_i \propto \sum_j A_{ij}\, f(\boldsymbol{u})_j = \big(A\, f(\boldsymbol{u})\big)_i\,.$$

Note that this relation is equivalent to the nonlinear eigenvector equation $Af(\boldsymbol{u}) = \lambda\boldsymbol{u}$ and, more in general, we can associate a *nonlinear eigenvector centrality*

$$\mathcal{M}_G(\boldsymbol{u}) = \lambda\boldsymbol{u}$$

to any nonnegative operator $\mathcal{M}_G : \mathbb{R}^n \to \mathbb{R}^n$ which exploits the graph topology in some sense.

Nonlinearity brings in two main advantages:

1. It allows us to incorporate into the centrality model a broad range of different (and possibly higher-order) topological properties;
2. It allows us to weaken the assumptions of the Perron–Frobenius theorem, in particular it allows us to drop the requirement on the nonzero entries of the graph matrix M_G.

For example, in the case of linear Bonacich centrality we have $M_G = A$ and the PF theorem would require the network to be strongly connected and aperiodic which is both (a) an expensive property to verify and (b) very rarely true in real-world data. Nonlinearity allows us to overcome these potential issues by relaxing the assumptions of the PF theorem. Precisely, the following result holds (the proof will be published in our next work [6] and it is partially based on [4])

[1] Precisely one needs M_G to be a primitive matrix, i.e. irreducible and such that $(M_G)^k > 0$ for some $k \geq 1$.

[2] We say that f is nonnegative if $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \geq 0$.

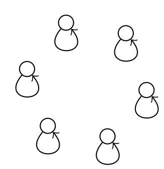
Theorem 1. *Let $\partial\mathcal{M}_G$ denote the Jacobian of $\mathcal{M}_G$. If $|\partial\mathcal{M}_G(\boldsymbol{x})|\boldsymbol{x} < \mathcal{M}_G(\boldsymbol{x})$ for all $\boldsymbol{x} > 0$, then there exists a unique nonlinear eigenvector centrality $\boldsymbol{u} > 0$, $\mathbb{1}^T\boldsymbol{u} = 1$ such that $\mathcal{M}_G(\boldsymbol{u}) = \lambda\boldsymbol{u}$ and we can compute $\boldsymbol{u}$ to an arbitrary precision ε in $O(c(\mathcal{M}_G)\log(1/\varepsilon))$ operations, where $c(\mathcal{M}_G)$ is the cost of applying $\mathcal{M}_G$.*

The talk will go deeper into this result by discussing several consequences in terms of optimization and application examples including the case of multilayer networks and nonnegative adjacency tensors. The following two examples help gaining further insight

Example 1

If $\mathcal{M}_G(\boldsymbol{x}) = Af(\boldsymbol{x})$ as in the nonlinear Bonacich model discussed above, the requirement of Theorem 1 boils down to $|f'(x_i)|x_i < x_i$. Note that this inequality does not hold for the linear case as $f(\boldsymbol{x}) = \boldsymbol{x}$ implies $f'(x_i) = 1$ and we have $|f'(x_i)|x_i = x_i$ in that case. In fact, we know that connectivity assumptions are required in that situation. However, if $f(\boldsymbol{x}) = \boldsymbol{x}^\theta$ for some $0 < \theta < 1$, we have $f'(x_i)x_i = \theta f(x_i)$ and we are guaranteed a unique nonlinear Bonacich centrality.

For example, in a very disconnected network with n isolated nodes, each having a self-loop, we have

$G =$ 

$$A = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

and any centrality vector is a Bonacich centrality, as $A\boldsymbol{u} = \boldsymbol{u}$, for all $\boldsymbol{u} > 0$. However, the equation $A\boldsymbol{u}^\theta = \lambda\boldsymbol{u}$ is satisfied only by $\boldsymbol{u} = \mathbb{1}$ and $\lambda = 1$, i.e. the unique nonlinear Bonacich centrality score is the "natural one", which assigns same score to all the isolated node.

Example 2

Suppose we are given a train transportation network. We have access only to the topology of connections between stations and we would like to identify highly populated stations. To this end, we may argue that a passenger prefers to use a station over another if it is well connected to important stations but at the same time it is surrounded by stations that are of minor relevance, in this way passengers are somewhat instinctively motivated to reach that station rather than using a less popular station in its neighborhood. So, if u_i is a measure accounting for

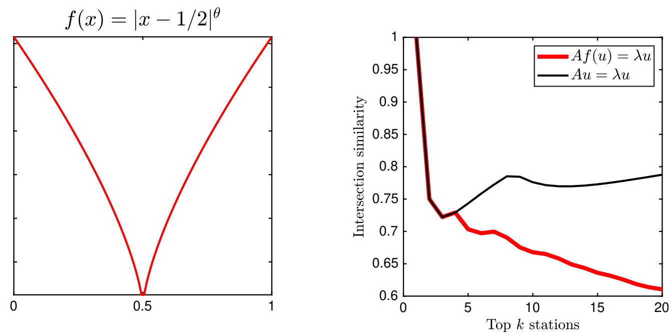


Figure 1: Left: Visualization of the test function f on a single variable $x \in \mathbb{R}$. Right: Intersection similarity between the sequence of stations sorted according to the actual number of passengers and according to the linear and nonlinear Bonacich centralities.

the popularity of a station, then u_i should grow if either many popular stations or many unpopular stations are in the neighborhood of i. To account for this we consider the "unsmoothed" modulus function $f(\boldsymbol{x}) = |\boldsymbol{x} - 1/2|^\theta$, $\theta \in (0, 1)$, and the associated nonlinear Bonacich centrality $Af(\boldsymbol{u}) = \lambda\boldsymbol{u}$. The advantages of this model are highlighted in Figure 1, where we show the intersection similarity (ISIM) between the the top twenty stations identified by $\boldsymbol{u}$ and the actual top-twenty most populated stations in London vs those identified by the linear Bonacich centrality. Recall that the lower is the ISIM the more the two sequences match. London rail and passenger data has been taken from [3, 2], respectively.

References

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