A MOTIF-BASED APPROACH TO PROCESSES ON NETWORKS: PROCESS MOTIFS FOR THE DIFFERENTIAL ENTROPY OF THE ORNSTEIN–UHLENBECK PROCESS

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Introduction

The study of motifs in networks has advanced the understanding of various systems in biology [2], ecology [7], economics [10], computational social science [4], and other areas. Traditionally, network scientists have considered graphlets (i.e., small subgraphs in a network) and identified them as motifs in a network's structure if empirical data [2] or mathematical models [5] indicate their importance to a system's function. We propose to consider "process motifs" (i.e., structured sets of walks) on a network as building blocks of processes on networks. We define a process motif on a graph G to be a directed weighted multigraph in which each edge corresponds to a walk in G and each edge's weight corresponds to the length of the associated walk. We demonstrate how one can derive process motifs for a property of noisy linear dynamics on networks using the steady-state differential entropy of the Ornstein–Uhlenbeck process as an example.

Differential entropy of the Ornstein–Uhlenbeck processes on a network

The Ornstein–Uhlenbeck process [13] is a simple and popular model for noisy coupled systems [1]. For example, it has been used as a model for the dynamics for neuronal systems [11], stock prices [6], and gene expression [8]. The differential entropy of the Ornstein–Uhlenbeck process at steady state is the basis for several properties of dynamical processes on networks. Examples of such properties are neural complexity [11], redundancy and degeneracy [12], and robustness to small perturbations in node states [3]. Across many domains, researchers have linked functions of entropy and differential entropy to a network's ability to robustly perform a desired function [3, 9]. Deriving process motifs that contribute to differential entropy of the Ornstein–Uhlenbeck process at steady state is a first step towards identifying the process motifs that contribute to neural complexity, redundancy, degeneracy, and other properties of processes on networks.

Processlets and emergence of process properties

For several properties of processes on networks, one can calculate the contribution of a process motif to the property. Studying the contributions of process motifs can further understanding of processes on networks in several ways. For example, it can help one understand how properties of processes on networks "emerge" from the superposition of small subprocesses. One can also calculate a graphlet's contribution to a property of a process from the contributions of process motifs that can occur on the graphlet. Researchers can then compare the contributions of different graphlets and rank graphlets by their contributions to a property of a process.

In Fig. 1, we show a graphlet and examples of associated process motifs. We also show several process motifs that contribute to steady-state differential entropy of the Ornstein–Uhlenbeck process and the respective contributions. We find that the process motifs that contribute to steady-state differential entropy are circular[1]. By considering different network structures, we find structures on which cyclic process motifs contribute most to differential entropy and structures on which acyclic process motifs contribute most to differential entropy.

Summary The analysis of process motifs and their contribution to differential entropy demonstrates that it can be useful to consider processes on a network (instead of just a network's structure) as a composite entity that one can decompose into many small parts. Such a decomposition of processes on networks provides a framework for studying the mechanisms by which processes and network structure contribute to differential entropy, redundancy, and other properties of processes on networks.

References

[1] O. O. Aalen and H. K. Gjessing. Survival models based on the Ornstein–Uhlenbeck process. *Lifetime Data Analysis*, 10(4):407–423, 2004.

[1] A *circular* process motif is a process motif such that if one replaces each directed edge by an undirected edge, the resulting graph is an undirected cycle.

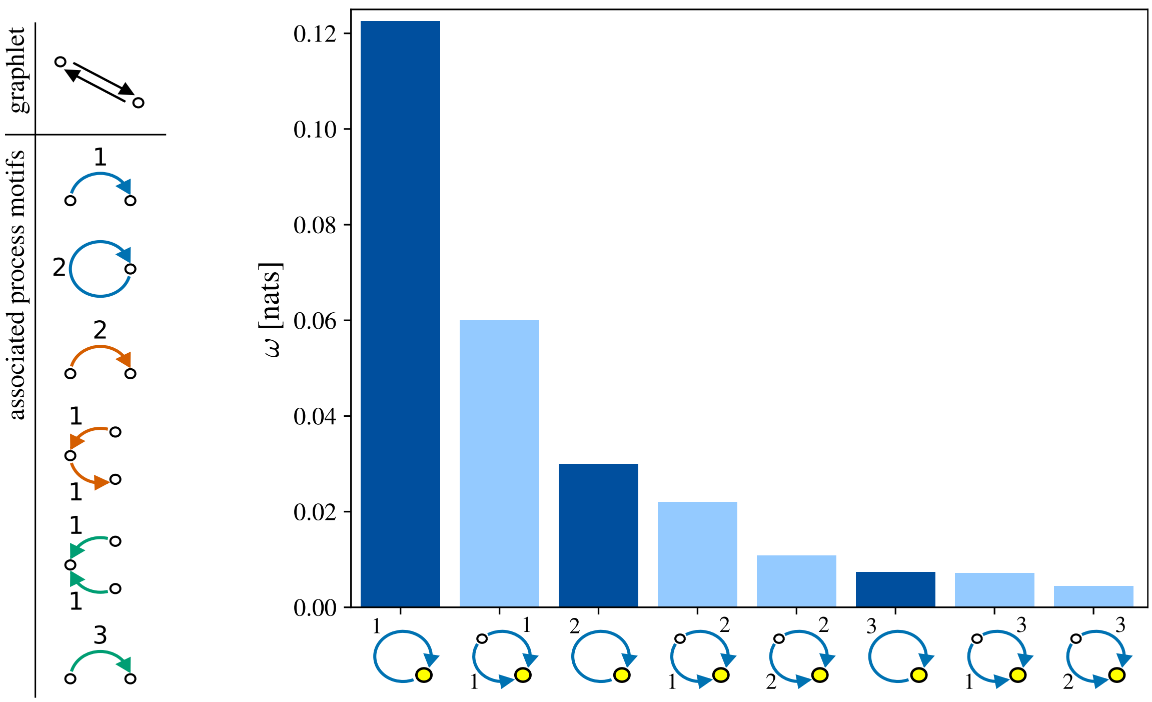

Figure 1: **Graphlets, process motifs, and process-motif contributions to differential entropy.** In the table on the left, we show a graphlet and examples of associated process motifs. Numerical labels indicate the length of an edge in a process motif. The process motifs with blue edges are examples of process motifs that use each edge in the graphlet at most once and in which each node corresponds to a different node in the graphlet. The process motifs with orange edges are examples of process motifs in which two nodes correspond to the same node in the graphlet. The process motifs with green edges are examples of process motifs that use edges in the graphlet more than once. In the plot on the right, we show contributions ω of process motifs to the steady-state differential entropy of the Ornstein–Uhlenbeck process. Bars are light blue when the corresponding process motif is acyclic and dark blue when the corresponding process motif is cyclic.

[2] U. Alon. Network motifs: Theory and experimental approaches. *Nature Reviews Genetics*, 8(6):450–461, 2007.

[3] L. Demetrius and T. Manke. Robustness and network evolution — An entropic principle. *Physica A: Statistical Mechanics and its Applications*, 346(3):682–696, 2005.

[4] X. Hong-Lin, Y. Han-Bing, G. Cui-Fang, and Z. Ping. Social network analysis based on network motifs. *Journal of Applied Mathematics*, (874708), 2014.

[5] J.-R. Kim, Y. Yoon, and K.-H. Cho. Coupled feedback loops form dynamic motifs of cellular networks. *Biophysical Journal*, 94(2):359–365, 2008.

[6] Z. Liang, K. C. Yuen, and J. Guo. Optimal proportional reinsurance and investment in a stock market with Ornstein–Uhlenbeck process. *Insurance: Mathematics and Economics*, 49(2):207–215, 2011.

[7] J. M. Rip, K. S. McCann, D. H. Lynn, and S. Fawcett. An experimental test of a fundamental food web motif. *Proceedings of the Royal Society of London B: Biological Sciences*, pages 1743–1749, 2010.

[8] R. V. Rohlfs, P. Harrigan, and R. Nielsen. Modeling gene expression evolution with an extended Ornstein–Uhlenbeck process accounting for within-species variation. *Molecular Biology and Evolution*, 31(1):201–211, 2013.

[9] T. A. Schieber, L. Carpi, A. C. Frery, O. A. Rosso, P. M. Pardalos, and M. G. Ravetti. Information theory perspective on network robustness. *Physics Letters A*, 380(3):359–364, 2016.

[10] F. W. Takes, W. A. Kosters, B. Witte, and E. M. Heemskerk. Multiplex network motifs as building blocks of corporate networks. *Applied Network Science*, 3(1):39, 2018.

[11] G. Tononi, O. Sporns, and G. M. Edelman. A measure for brain complexity: Relating functional segregation and integration in the nervous system. *Proceedings of the National Academy of Sciences of the United States of America*, 91(11):5033–5037, 1994.

[12] G. Tononi, O. Sporns, and G. M. Edelman. Measures of degeneracy and redundancy in biological networks. *Proceedings of the National Academy of Sciences of the United States of America*, 96(6):3257–3262, 1999.

[13] G. E. Uhlenbeck and L. S. Ornstein. On the theory of the Brownian motion. *Physical Review*, 36(5):823–841, 1930.